Stable localized modes in asymmetric waveguides with gain and loss

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It is shown that asymmetric waveguides with gain and loss can support a stable propagation of optical beams. This means that the propagation constants of modes of the corresponding complex optical potential are real. A class of such waveguides is found from a relation between two spectral problems. A particular example of an asymmetric waveguide, described by the hyperbolic functions, is analyzed. The existence and stability of linear modes and of continuous families of nonlinear modes are demonstrated.   © 2014 Optical Society of America

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Waveguides are characterized by a specific variation of the refractive index (RI) in the transverse direction. Usually the refractive index at a central (core) region is larger than that at cladding. Such variation creates an optical potential that supports eigenmodes with real values of the propagation constant. Each eigenmode corresponds to the field distribution of a stable beam that can propagate in a waveguide.

Even small loss (gain) destroys waveguide modes, resulting in a decrease (increase) of the mode amplitude. In [1,2], it is shown that waveguides with a proper distribution in the transverse direction of gain and loss can have stable eigenmodes. The idea of these works comes from attempts (for example see [3]) to generalize quantum mechanics to complex potentials. It was found that a non-Hermitian Hamiltonian with a complex potential, satisfying the parity-time (PT) symmetry, can have a real spectrum. A PT-symmetric potential $V(x)$ complies with the following condition: $V(x) = V^*(-x)$, where a star sign means a complex conjugate.

In the context of optics, the PT-symmetry means that the real (imaginary) part of the RI is an even (odd) function of x. A study of PT-symmetric optical structures is now an active field of research [1,2,4–10].

In present Letter, we extend a class of complex potentials that admit a real spectrum. Namely, we present a class of *asymmetric* potentials (waveguides) that support stable localized modes. There are other examples of complex waveguides without PT-symmetry that have a real spectrum. For example, in [9], such waveguides are found using the supersymmetry (SUSY) method of quantum mechanics. The SUSY method [11–13] allows one to construct two potentials that have a related spectrum, using the same spectral problem. In contrast, the approach used in this Letter is based on a relation between two different spectral problems, namely, the Schrödinger problem and the Zakharov–Shabat problem.

We start with the general nonlinear Schrödinger (NLS) equation that describes in the parabolic approximation the propagation of optical beams in waveguides [14].

$$i\psi_z + \psi_{xx}/2 + V(x)\psi + \gamma|\psi|^2\psi = 0, \qquad (1)$$

where $\psi(x,z)$ is an envelope of the electric field and x and z are transversal and longitudinal coordinates: Equation (1) is written in a dimensionless form. Variable z is normalized with $Z_s = k_0 n_0 X_s^2$, while the field amplitude scale is $\Psi_s = 1/[(n_0 n_2)^{1/2} k_0 X_s]$, where X_s is the size of a beam in the transverse direction, n_0 is the background refractive index, n_2 is the nonlinear (Kerr) coefficient, $k_0 = 2\pi/\lambda$, and λ is the laser wavelength. We consider the RI as $n(x,z) = n_0 + \Delta n(x) + n_2 I(x,z)$, where $I(x,z)$ is the beam intensity. Potential $V(x) = V_R(x) + iV_I(x)$ is related to the variation $\Delta n(x)$ of the RI, $V(x) = k_0^2 n_0 X_s^2 \Delta n(x)$. Parameter $\gamma = 0$ ($\gamma = \pm 1$) for the linear (nonlinear) case.

In order to find eigenmodes of Eq. (1), the field is represented as $\psi(x,z) = u(x)\exp(i\mu z)$, where $u(x)$ is a stationary mode and μ is the propagation constant. The mode $u(x)$ and μ are found from the following spectral problem:

$$u_{xx}/2 + V(x)u + \gamma|u|^2 u = \mu u. \qquad (2)$$

The boundary condition is $u(\pm\infty) = 0$, since we are interested in localized modes.

We consider first the linear case, $\gamma = 0$. Then Eq. (2) corresponds to the stationary Schrödinger equation with potential $V(x)$. When complex potential $V(x)$ is taken in the following form:

$$V(x) = [v^2(x) \pm iv_x(x)]/2, \qquad (3)$$

where $v_x \equiv dv/dx$, then Eq. (2) is related to another spectral problem, namely to the Zakharov–Shabat (Z–S) problem [15,16]. The Z–S problem is defined for a two-component vector (ϕ_1, ϕ_2) and the spectral parameter λ as the following:

$$i\phi_{1,x} - iv(x)\phi_2 = \lambda\phi_1,$$
$$-i\phi_{2,x} - iv^*(x)\phi_1 = \lambda\phi_2. \qquad (4)$$

If we take [4,16],

$$u = \phi_1 + i\phi_2, \qquad r = -i(\phi_1 - i\phi_2), \qquad (5)$$

then an equation for $u(x)$ is reduced to Eq. (2) with $V(x)$ defined in Eq. (3) with the plus sign and

$$\mu = -\lambda^2/2. \qquad (6)$$

An equation for $r(x)$ differs by sign in front of v_x, which is why the sign $\pm$ is taken in Eq. (3). In general, potential $v(x)$ is a complex function in the Z–S problem. However, in this Letter, we consider only real $v(x)$, since the relation between Eqs. (2) and (4) is valid only in this case. Transformation (5) is well known (e.g., see [16], Sec. 2.12 and [4,12]), but to the best of our knowledge, there is no application of this result to waveguides with gain and loss.

The relation between the two spectral problems results in an important conclusion that if the discrete spectrum λ_j of the Z–S problem (4) with a real potential $v(x)$ is purely imaginary, then the Schrödinger problem (2) with potential (3) has a *real* spectrum μ_j, found from Eq. (6), where $j = 1, 2, \ldots$. Since $V_I(x) \sim v_x$, then $\int_{-\infty}^{\infty} V_I(x)\mathrm{d}x = 0$ if $v(\pm\infty) \to 0$, so that complex waveguides with potential (3) are gain–loss balanced waveguides.

If $v(x)$ is an even function, then $V(x)$ in (3) is a PT-symmetric potential. Moreover, if the corresponding Z–S problem has a purely imaginary discrete spectrum, then the parameters of the complex potential $V(x)$ are below the PT-symmetry breaking threshold. One example of a potential with a purely imaginary spectrum is a rectangular box [17]. Another example is the potential $v(x) = v_0 \operatorname{sech}(x)$ [18]. Therefore, waveguides with gain and loss, corresponding to these potentials, have stable localized modes. The PT-symmetric potential $V(x)$, found from Eq. (3) with $v(x) = v_0 \operatorname{sech}(x)$, is studied in [1,19].

It is necessary to note that the Z–S problem (4) with a real potential has, in general, complex eigenvalues (EVs) [15,20,21]. Let us consider a single-hump potential $v(x)$, i.e., a function that is nondecreasing (nonincreasing) on the left of some $x = x_p$ and nonincreasing (nondecreasing) on the right of x_p. As demonstrated in [20], the Z–S problem with a single-hump potential has a purely imaginary discrete spectrum. From this result and the relation between the two spectral problems (2) and (4), we infer that if $v(x)$ is a single hump (*asymmetric*, in general) function, then the Schrödinger equation (2) with potential (3) has real EVs. This means that a waveguide with a single-hump distribution of the real part of the refractive index and with the distribution of gain and loss defined by (3) has stable localized modes.

As an example, we consider an asymmetric potential $V(x)$ defined as

$$V(x) = \frac{1}{2}\left[\eta v_0^2 \operatorname{sech}^2(x/w) - i\frac{v_0}{w} \operatorname{sech}(x/w)\tanh(x/w)\right], \qquad (7)$$

where

$$w = w_1 \quad \text{for } x < 0, \quad \text{and} \quad w_2 \quad \text{for } x \geq 0, \qquad (8)$$

and v_0, w_1, w_2 and η are constant parameters. When $\eta = 1$, the potential (7) corresponds to form (3), where
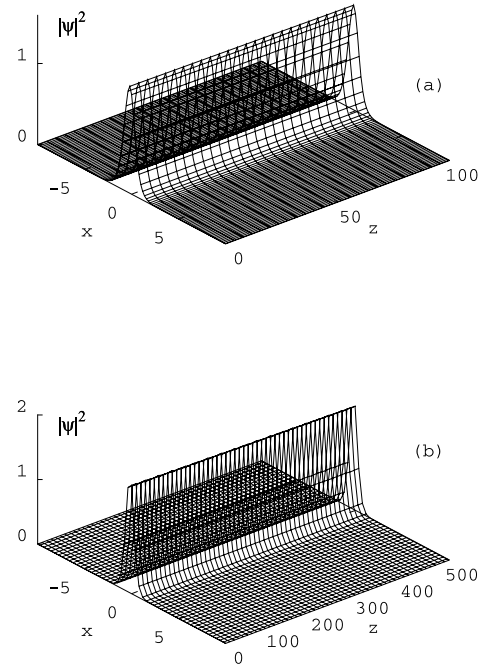

Fig. 1.   Stable dynamics of waveguide modes for (a) $\gamma = 0$, and (b) $\gamma = 1$ and $P = 2$. The other parameters are $v_0 = 2$, $w_1 = 1$, and $w_2 = 0.5$.

$v(x) = v_0 \operatorname{sech}(x/w)$. We use this value of η in all numerical examples below.

Figure 1(a), which is a result of numerical simulations of Eq. (1), demonstrates the stable propagation of a waveguide mode for $v_0 = 2$, $w_1 = 1$ and $w_2 = 0.5$. As an initial condition, we use an exact eigenmode found numerically from Eq. (2). Since the potential $V(x)$ is asymmetric, the corresponding localized modes are also asymmetric.

Figure 2 shows the dependence of discrete EVs μ_j on the potential parameter v_0. The EVs are found from the numerical solution of the spectral problem (2) with the potential (7) and (8). We apply the shooting method (e.g., see [22]), integrating Eq. (2) from the left and from the right in a sufficiently large interval of x. A mismatch of the function u and its derivative at a fitting point is

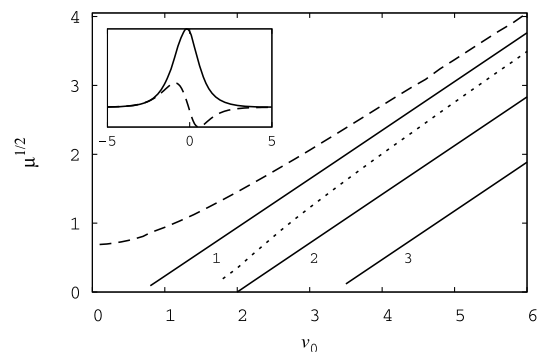
Fig. 2.   Dependence of EVs (propagation constant) μ_j on v_0 for potential (7), $w_1 = 1$ and $w_2 = 0.5$. Solid (dashed and dotted) lines are for the linear (nonlinear) modes, $\gamma = 0$ ($P = 2$, $\gamma = 1$ and $\gamma = -1$). The inset shows the real part (solid line) and the imaginary part (dashed line) of the fundamental eigenmode of a linear waveguide for $v_0 = 2$.

minimized by an optimization routine via an adjustment of the boundary conditions [22].

Only the first three EVs are shown in Fig. 2. A number near a curve corresponds to the number of the EV, $j = 1$, 2 and 3. It is known that for symmetric cases, $w_1 = w_2$, the dependence of the EV λ_j of the Z–S problem is linear on v_0 [18]; therefore, μ_j is quadratic on v_0. As follows from Fig. 2, such quadratic dependence remains in the asymmetric case as well.

Value $\eta \neq 1$ introduces imbalance between the real and imaginary parts of potential (7). In this case, there is no corresponding Z–S problem with a real potential. As a result, a spectrum of Eq. (2) with potential (7) is complex. For example, for $u_0 = 2$, $w_1 = 1$ and $w_2 = 0.5$, when $\eta = 0.8$, $\mu_1 = 0.57 + 0.0082i$, and when $\eta = 1.2$, $\mu_1 = 1.2 - 0.0039i$.

Now we consider the nonlinear case, $\gamma = \pm 1$. As follows from Eq. (1), nonlinearity results in an additional self-induced potential $V_{\mathrm{NL}} = \gamma|\psi|^2$. Though V_{NL} is added only to the real part of V, the effect of nonlinearity is not the same as the deviation of η from the unity discussed above. Namely, we find that the spectrum of the nonlinear system remains real in a wide range of the parameters.

In presence of nonlinearity, EVs depend on the beam amplitude (or on the total power $P = \int_{-\infty}^{\infty} |\psi|^2 dx$) as well. The dependencies of the ground state EVs on the amplitude v_0 for both values of γ and $P = 2$ are shown by dashed and dotted lines in Fig. 2. The value $P = 2$ corresponds to the total power of the fundamental soliton with unit amplitude that exists when $V(x) = 0$ and $\gamma = 1$. For $\gamma = 1$ ($\gamma = -1$), nonlinearity induces an attractive (repulsive) potential, resulting in an increase (decrease) of nonlinear EVs comparing with the linear ones.

The nonlinear EVs in Fig. 2 are found from Eq. (2) by the shooting method. The optimization routine looks for a continuous and smooth mode with the specified value of P. In all cases studied, the imaginary part of μ is of the order of the accuracy (typically 10^{-7}–10^{-9}) of the optimization routine. By increasing the range of x and the accuracy, Im[μ] can be further decreased below 10^{-10}.

The propagation of the nonlinear mode is shown in Fig. 1(b). As an initial condition, we use an exact eigenfunction multiplied by $[1 + \epsilon(x)]$, where $\epsilon(x)$ is a random field with the uniform distribution in the range $[0, \ 0.01]$. Absorbing boundary conditions are implemented to minimize reflection of linear waves from the ends of the computation window. The stable dynamics of nonlinear modes, as in Fig. 1(b), is observed also for other values of the system parameters and the soliton power.

It is necessary to mention that, actually, the mode power slightly decreases in Figs. 1(a) and 1(b), and can be approximated as $P(z) \approx P_0 \exp(-\delta z)$, where $\delta \sim 10^{-4} - 10^{-6}$. However, this decrease is due to the mode asymmetry and numerical discretization. It follows from Eq. (1) that

$$P_z = -\int_{-\infty}^{\infty} \mathrm{Im}[\psi_{xx}\psi^*]dx - \int_{-\infty}^{\infty} 2V_I|\psi|^2 dx \equiv -I_1 - I_2. \quad (9)$$

Integral I_1 in Eq. (9) vanishes if one calculates it analytically. This integral, calculated numerically with

equidistant discretization of asymmetric ψ on a finite range of x, is not zero. Since the numerical error of calculation of ψ_{xx} is of the order of $(\Delta x)^2$, where Δx is a step on x, integral I_1 can be estimated as $\sim (\Delta x)^2 P$. Therefore, the rate $\delta \sim (\Delta x)^2$ can be varied by the step Δx. Indeed, from numerical simulations of Eq. (1) for different initial conditions, we find that $\delta \approx 2.3 \cdot 10^{-4}$ for $\Delta x = 0.024$ and $\delta \approx 1.8 \cdot 10^{-5}$ for $\Delta x = 0.006$. The dependence of δ on Δx indicates that the decrease of P is the result of the numerical procedure, rather than the complex value of μ. For symmetric potentials, $w_1 = w_2$, the soliton power is constant in numerical simulations. Integral I_2 is zero when calculated numerically on a mode for symmetric and asymmetric potentials.

The spectrum of nonlinear modes is real for other values of P (see Fig. 3). The propagation constant of solitons tends to that of the linear mode for either sign of γ when $P \to 0$. Therefore, there are continuous families of stationary solitons that bifurcate from linear modes.

An existence of continuous families of solitons in asymmetric complex potentials is an important observation. Typically, for a given set of the system parameters, dissipative solitons exist only for a fixed value(s) of the soliton amplitude (power) (e.g., see [10,23]). In a recent paper [10], the author suggests that PT-symmetry is "a necessary condition for the existence of soliton families" in the NLS equation with a complex potential. However, this statement is based on the perturbation approach and is not proven rigorously [10]. Our results indicate that this condition should be extended to a wider class of potentials.

Figure 4 shows the dependence of the amplitude A (the maximum of $|\psi(x, z)|$ on x) and the full width at the half-maxima (FWHM) a of the nonlinear fundamental modes (cf. Fig. 2) on the potential parameter v_0. The amplitude of the mode for focusing nonlinearity ($\gamma = 1$) is larger than that for defocusing nonlinearity ($\gamma = -1$), while there is an opposite relation for the mode width. It is known that the amplitude A_s and the width a_s of the fundamental soliton of the standard NLS model, Eq. (1) with $V(x) = 0$ and $\gamma = 1$, are related to each other, namely, $A_s a_s \approx 1.76$ (e.g., see [14]). In contrast, the product Aa for the fundamental nonlinear mode varies on v_0.
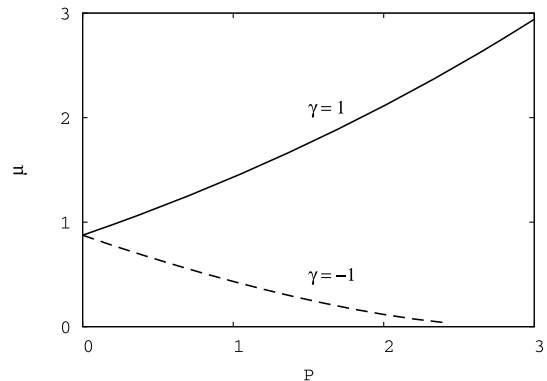

Fig. 3. Dependence of μ on P for $\gamma = 1$ (solid line) and for $\gamma = -1$ (dashed line). The parameters of potential (7) are $v_0 = 2$, $w_1 = 1$, and $w_2 = 0.5$.
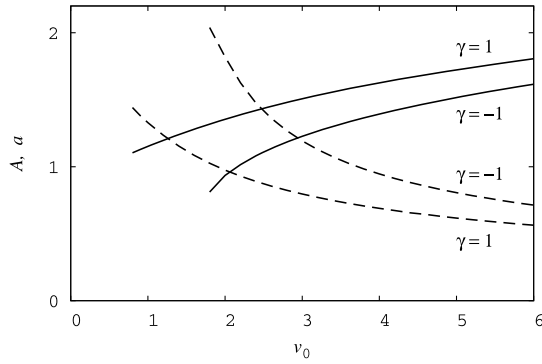
Fig. 4. Dependence of the amplitude A (solid lines) and of the FWHM a (dashed lines) of the nonlinear fundamental mode on v_0 for potential (7), $w_1 = 1$, $w_2 = 0.5$ and $P = 2$.

We also mention that the Z–S spectral problem is associated via the inverse scattering transform method with the standard NLS equation [15,16]:

$$iq_z + q_{xx}/2 + |q|^2 q = 0. \qquad (10)$$

This establishes a relation between the Schrödinger equation (2), $\gamma = 0$, with the complex potential (3) and the NLS equation (10). In particular, if an initial condition, $q(x, 0) = v(x)$, where $v(x)$ is a real function, of Eq. (10), results in nonmoving solitons only, then the discrete spectrum of Eq. (2) with (3) is pure real.

In conclusion, in this Letter, we demonstrate how the relation between two spectral problems helps to design waveguides with gain and losses that possess a real spectrum. It is shown that a waveguide with the distribution of the RI in form (3), where $v(x)$ is any (symmetric or asymmetric) single-hump function, has a real spectrum and, therefore, stable localized modes. A particular example of asymmetric optical potential (7) with real EVs is considered. It is also shown that continuous families of stable localized modes can exist in nonlinear waveguides with asymmetric distribution of the RI.

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